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**AN INTEGER LINEAR PROGRAMMING APPROACH
TO THE VEHICLE SCHEDULING PROBLEM**

by

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ABSTRACT

This paper considers a version of the vehicle scheduling problem (VSP) in which a non-negative weight is assigned to each city to be visited and where all vehicles are identical and have the same capacity. The weight assigned to a vehicle on a given route may not exceed this capacity. Cities can be visited more than once if it is economical to do so; however, some cities (with zero weights) may not be visited. This paper describes an optimal integer linear programming approach to the VSP, based on earlier work by Fleischmann and by the author.

1. INTRODUCTION

We consider the vehicle scheduling problem (VSP) as defined by Fleischmann [3] in a recent paper: let $G = (V, E, C)$ be a symmetric network where

- V is the set of vertices,
- E is the set of (undirected) edges,
- $C = (c_{ij})$ is the (symmetric) matrix of distances between the vertices.

The vertices represent cities or clients to be visited by vehicles, the edges roads. We allow the case where only a subset V_0 of vertices (called "visiting vertices") V_0 of V have to be visited. Let us also define

- D_i the (non-negative) demand at city i ($i \in V_0$)
- K the maximum capacity of a vehicle,
- $i_0 \in V_0$ the depot, with $D_{i_0} = 0$. (In this paper, $i_0 = 1$.)

The VSP consists of establishing a set of routes originating from the depot such that

- (i) the total distance travelled is minimized;
- (ii) the capacity of a vehicle on any route is never exceeded;
- (iii) each city of V_0 is visited at least once.

When $V_0 = V$, this problem reduces to

- (TSP) the travelling salesman problem (see for example [1]) if only one vehicle must be used, all D_i 's are equal to zero and each city must be visited exactly once;
- (m-TSP) the m-travelling salesman problem (see for

example [6]) if m vehicles are used, all D_i 's are equal to zero and each city (except i_0) must be visited exactly once;

(CVRP) the capacity constrained vehicle routing problem (see for example [7]) if conditions (i) and (ii) hold, but each city (except i_0) must be visited exactly once.

In the three above cases, a feasible solution may not exist if the graph G is not complete, i.e. if an edge is not defined for each pair of vertices. However, if only condition (ii) is lifted (i.e. all D_i 's are equal to zero), the VSP reduces to

(CCP) the problem of finding the shortest complete cycle in a graph (see for example [8]).

Finally, the case where all D_i 's are equal to zero but where $V_0 \neq V$ has been treated in [9]. It is known as

(RP- k) the routing problem with k specified nodes ($k = |V_0|$).

The VSP defined by Fleischmann is therefore more general than any of the above five problems. It is also more realistic than the standard Vehicle Routing Problem (see for example [7]) treated in the literature since in practice not all clients of V always have to be visited; furthermore, road networks do not usually contain a direct link between each pair of cities and it may therefore prove necessary to go through the same city more than once; finally, even if there exists a solution in which each city is visited exactly once, it may be preferable from an economic point of view to allow multiple visits.

Fleischmann solves the VSP by first considering one of its relaxations called the VSP'. When an optimal solution to the VSP' has been identified, it may constitute a feasible (and optimal) solution to the VSP if all constraints included in the VSP but

not in the VSP' are satisfied. But this is not always the case: out of six problems solved by Fleischmann, two instances occur where the optimal VSP' solution does not satisfy all VSP constraints. Our aim is to show how VSP feasibility can be reached by introducing a suitable constraint to an optimal VSP' that does not satisfy all of the VSP constraints. By generating a sufficient number of this type of constraint, a feasible and optimal solution for the VSP can be reached in a finite number of steps.

2. FLEISCHMANN'S FORMULATION FOR THE VSP'

As mentioned at the end of the last section, Fleischmann solves a relaxation of the VSP called the VSP'.

Let

x_{ij} = the number of times the optimal tour passes through edge (i,j) ($i < j$) of E in the optimal solution

$[t]$ = $\begin{cases} \text{the smallest integer greater than or equal to } t & (t > 0) \\ 1 & (t = 0) \end{cases}$

$\bar{W} = V - W \quad (W \subseteq V)$

Then the VSP' can be formulated as follows:

(VSP') minimize $\sum_{i < j} c_{ij} x_{ij}$

subject to

$$(1) \quad \sum_{i < k} x_{ik} + \sum_{j > k} x_{kj} = 2y_k \quad (k \in V)$$

$$(2) \quad \sum_{\substack{i \in W, j \in \bar{W} \\ \text{or } i \in \bar{W}, j \in W}} x_{ij} \geq 2 \left[\frac{\sum_{i \in W} D_i}{K} \right] \quad (W \subseteq V - \{1_0\}, W \cap V_0 \neq \emptyset)$$

- (3) x_{ij} non-negative and integer $(i < j)$
- (4) $y_k \geq 1$ and integer $(k \in V_0)$
- (5) $y_k \geq 0$ and integer $(k \in \bar{V}_0)$

In this formulation, constraints (1) and (4) ensure that each vertex of V_0 is visited at least once and that its degree is even; similarly, constraints (1) and (5) ensure that all vertices of $\bar{V}_0$ are even. Constraints (2) are those used for the CVRP: they guarantee that every set W of nodes is connected to the depot in the optimal solution and is visited by at least $\left\lceil \frac{\sum_{i \in W} D_i}{K} \right\rceil$ vehicles.

This formulation, as we shall see later, does not necessarily provide a feasible solution to the VSP. Before examining this point, we make the following observations:

- (i) let G^* be the graph obtained by repeating x_{ij} times each edge (i,j) of G ; since G^* contains no disconnected set of nodes (apart from possibly some nodes from $\bar{V}_0$ which need to be visited) and since the degree of each node is even, it possesses a Euler cycle [2];
- (ii) as noted in [8], if all D_i 's are equal to zero, constraints (3) can be replaced by

$$(3') \quad x_{ij} = 0, 1, 2 \quad (i < j)$$

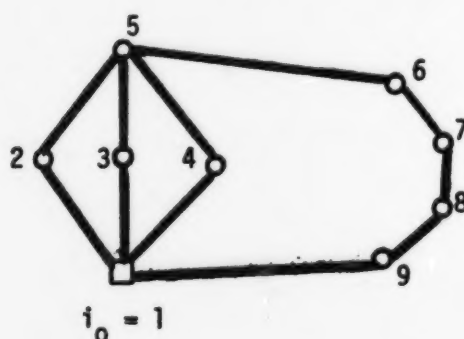
However, this is not true in general. For example, if only one edge is incident to i_0 , it will be crossed more than twice whenever $\sum_{i \in V_0} D_i > K$.

Fleischmann rightly observes that unless all degrees (except that of i_0) are equal to 2 in the optimal solution, not every solu-

tion to the VSP' is feasible for the VSP. He gives the following counter-example (see figure 1): suppose that all D_i 's are equal to 1, that $K = 4$ and that all edges shown in the figure have an associated x_{ij} equal to 1. The example clearly satisfies all constraints of the VSP': even degrees, integrality of the variables and connectedness. Constraints (3) are satisfied for each subset W of nodes. But "the solution cannot be split into two tours with load 4". It does therefore happen that the optimal solution to the VSP' is not feasible for the VSP. In the next section, we show how to circumvent this problem.

FIGURE 1

A solution which is feasible for the VSP', but not for the VSP



3. OBTAINING A FEASIBLE AND OPTIMAL SOLUTION TO THE VSP

We propose first solving the VSP' by means of the constraint relaxation method developed by Fleischmann, which can be summarized as follows:

- (i) optimize a first problem containing all degree (type (1)) constraints;
- (ii) if all type (2) constraints are satisfied, go to step (iii); introduce all violated type (2) constraints; reoptimize and repeat this step until all type (2) constraints are satisfied;

- (iii) if the solution is integer, go to step (v); otherwise, try to identify and generate "three-star constraints" [3]; if no such constraint can be generated, go to step (iv); reoptimize and go to step (ii);
- (iv) introduce a Gomory cut [4]; reoptimize and go to step (ii);
- (v) a feasible and optimal solution for the VSP' has been found; the algorithm terminates.

If the solution identified in step (v) is feasible for the VSP, then it is also optimal for the VSP. Otherwise, it can be eliminated by the introduction of a constraint that removes only the current (VSP) infeasibility; the algorithm would then proceed to step (ii) and terminate only when VSP feasibility is reached.

In order to derive such a constraint, we first observe that in an optimal linear program of the form

$$\begin{aligned}
 \text{(LP)} \quad & \text{minimize} \quad cx \\
 & \text{subject to} \quad Ax \geq b \\
 & \quad \quad \quad x \geq 0
 \end{aligned}$$

one obtains a valid constraint $g(x) \geq h$ whose left hand side is minimized at the current optimal solution, by adding non-negative weights of the effective (basic) constraints at the optimum. (Gomory cuts are generated according to this principle, by selecting the weights so as to obtain a linear function of integer coefficients minimized at a fractional value [4]). If the weights are strictly positive, $g(x)$ is minimized *uniquely* at the current optimal point, i.e. if h^* denotes the minimum, the intersection of the hyperplane $g(x) = h^*$ with the feasible region consists of only one point. Thus, in an optimal pure integer linear program, one can generate a valid constraint which cuts off only the current optimum, by adding strictly positive multiples of the effective constraints (including effective upper bound and non-negativity constraints) and by adding 1 to the

right hand side. This is the procedure which was applied in [5] to eliminate from an integer linear program an optimal solution which did not satisfy some constraints not explicitly stated. In that case, all weights were set equal to 1.

It would therefore be possible to introduce such a procedure in step (v) of Fleischmann's algorithm (and then reoptimize and go to step (ii)); the process would then converge to a solution that is feasible and optimal to the VSP.

It is however possible (and desirable) to increase the rate of convergence by making use of the following observation. Consider the optimal VSP' solution at step (v). It is integer and consists of k connected components $G_1, \dots, G_k$ having a unique common vertex i_0 (a connected component is a subgraph $G_i = (V_i, E_i, C_i)$ of G such that $x_{ij} > 0$ for all edges (i,j) of E_i). Let $VSP'(G_i)$ be the VSP' defined on G_i (hence $VSP' = VSP'(G)$). Define $VSP(G_i)$ similarly. Since $VSP'(G)$ is optimal at the current solution, so are $VSP'(G_1), \dots, VSP'(G_k)$. The procedure just described can be applied to each connected component G_i such that the solution to $VSP'(G_i)$ is not feasible for $VSP(G_i)$. This way of proceeding is clearly stronger than just generating one new constraint for the whole problem (VSP') since it guarantees that each connected component that is infeasible for the VSP will not reappear at a later stage.

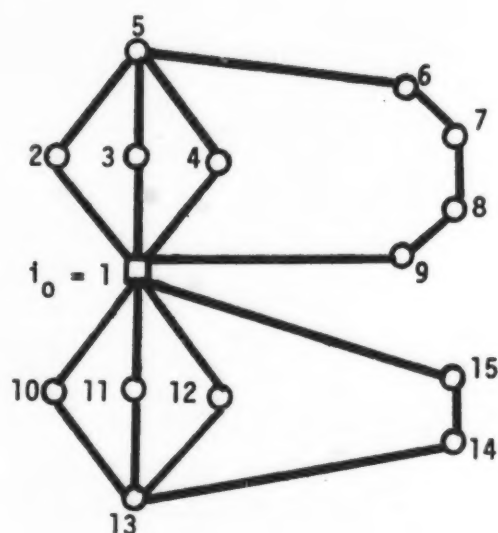
For example, consider the situation represented by figure 2. As in figure 1, all D_i 's are equal to 1 and $K = 4$.

The graph represented in figure 2 consists of two connected components G_1 and G_2 (with respective vertex sets $V_1 = \{1,2,3,4,5,6,7,8,9\}$ and $V_2 = \{1,10,11,12,13,14,15\}$) having vertex 1 in common. As indicated earlier (see figure 1), component G_1 is infeasible for $VSP(G_1)$ whereas G_2 is feasible for $VSP(G_2)$ since its vertices

can be covered by the two feasible tours (1-10-13-11-1) and (1-12-13-14-15-1). In this case, it would only be necessary to generate a new constraint for G_1 .

FIGURE 2

Optimal solution to the VSP' consisting of two connected components



It should finally be noted that identifying which constraints are effective in the optimal solution of $VSP'(G_1), \dots, VSP'(G_k)$ is a relatively simple task since all the information required is provided by the optimal solution of $VSP'(G)$.

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